Math 210B Lecture 23 Notes

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1 Discriminants of Linear Maps

1.1 Hilbert's theorem 90

Let's complete our proof of Hilbert's theorem 90.

Theorem 1.1 (Hilbert's theorem 90)**.** *Let E/F be finite, Galois with cyclic Galois group $G = \langle \sigma \rangle$. Then*

$$\ker(N_{E/F}) = \{\sigma(x)/x : x \in E^\times\},$$

$$\ker(\mathrm{tr}_{E/F}) = \{\sigma(x) - x : x \in E\}.$$

Last time, we proved the result for the trace.

Proof. $\dim \ker(\mathrm{tr}) \geq n - 1$, where $n = [E : F]$. Since $\ker(\mathrm{tr}_{E/F}) \supseteq \{\sigma(x) - x : x \in E\}$, it suffices to show that $\mathrm{tr}_{E/F} \neq 0$. Write the trace as $\mathrm{tr}_{E/F} = \sum_{\sigma \in G} \sigma$. This is a nonzero linear combination of characters, so $\mathrm{tr}_{E/F} \neq 0$. □

1.2 Discriminants of linear maps

Recall that if $f \in F[t]$ factors in $\overline{F}$ as $f = \prod_{i=1}^n (t - \alpha_i)$, then the discriminant is $\mathrm{disc}(f) = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$. If $F(\alpha) = E/F$ is Galois and f is the minimal polynomial of α, then we can embed $G \to A_n$ iff $\mathrm{disc}(f)$ is a square in F.

Let V be an F-vector space with $\dim(V) = n$. The space $\{\psi : V \otimes V \to F\}$ of bilinear forms on V has dimension n^2. Let $\beta = \{v_1, \dots, v_n\}$ be an ordered basis for V. Then

$$\mathrm{Hom}(V \otimes_F V, F) \cong M_n(F),$$

via the maps

$$\psi \mapsto M_\psi = [\psi(v_i \otimes v_j)]_{i,j},$$

$$\psi_M(v_i \otimes v_j \mapsto v_i^\top M v_j) \leftarrow M.$$

Definition 1.1. The **discriminant** of ψ (with respect to β) is $\mathrm{Disc}_\beta(\psi) = \det(M_\psi)$.

Proposition 1.1. *Let* $T : V \to V$ *be linear with basis* β *of* V*. Let* $T \otimes T : V \otimes V \to V \otimes V$*. Then*

$$\mathrm{Disc}_\beta(\psi \circ T \otimes T) = \det(T)^2 \, \mathrm{Disc}_\beta(\psi).$$

Proof. $\psi(Tv_i, Tv_j) = ([T]_\beta, e_i)^\top M_\psi [T]_\beta e_j$, so

$$M_{\psi \circ T \otimes T} = [T]_\beta^\top M_\psi [T]_\beta.$$

□

Let E/F be a field extension, and let $\beta = \{v_1, \dots, v_n\}$ be a bassi for E/F. Let

$$E \otimes E \xrightarrow{m} E \xrightarrow{\mathrm{tr}_{E/F}} F$$

send $v \otimes W \mapsto \mathrm{tr}(vw)$. Call this composition map tr.

Proposition 1.2. *Let* $\mathrm{Emb}_F(E) = \{\sigma_1, \dots, \sigma_n\}$*. Define* $Q = [\sigma_i(v_j)]_{i,j}$*. Then* $M_{\mathrm{tr},\beta} = Q^\top Q$*. In particular,*

$$\mathrm{Disc}_\beta(\mathrm{tr}) = \det(Q)^2.$$

Proof.

$$\begin{aligned} \mathrm{tr}(v_i, v_j) &= \sum_{k=1}^n \sigma_k(v_i v_j) \\ &= \sum_{k=1}^n \sigma_k(v_i)\sigma_k(v_j) \\ &= (Q^\top Q)_{i,j}. \end{aligned}$$

□

Let $f(t) = \prod_{i=1}^n (t - \alpha_i) \in F[t]$ be irreducible and separable. Consider $F(\alpha_1)/F$. We have the nice basis $\beta = \{1, \alpha_1, \dots, \alpha_1^{n-1}\}$. THen $\mathrm{Emb}_F(F(\alpha)) = \{\sigma_i : \alpha_1 \mapsto \alpha_i\}$. Then

$$Q(\alpha_1, \dots, \alpha_n) = \begin{bmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \cdots & \alpha_n^{n-1} \end{bmatrix}$$

is the **Vandermonde matrix**.

Proposition 1.3. $\det(Q(\alpha_1, \dots, \alpha_n)) = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i)$.

Proof.

$$\begin{vmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \cdots & \alpha_n^{n-1} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & \alpha_2 - \alpha_1 & \cdots & \alpha_2^{n-2}(\alpha_2 - \alpha_1) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n - \alpha_1 & \cdots & \alpha_n^{n-2}(\alpha_n - \alpha_1) \end{vmatrix}$$
$$= 1 \begin{vmatrix} \alpha_2 - \alpha_1 & \cdots & \alpha_2^{n-2}(\alpha_2 - \alpha_1) \\ \vdots & \vdots & \vdots \\ \alpha_n - \alpha_1 & \cdots & \alpha_n^{n-2}(\alpha_n - \alpha_1) \end{vmatrix}$$
$$= (\alpha_2 - \alpha_1) \begin{vmatrix} 1 & \alpha_2 & \cdots & \alpha_1^{n-2} \\ 1 & \alpha_3 & \cdots & \alpha_2^{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \cdots & \alpha_n^{n-2} \end{vmatrix}.$$

This is the Vandermonde determinant for $n-1$ variables. By induction, we are done. □

So if $F(\alpha)/F$ is eparable and f is the minimum polynomial of α, then

$$\mathrm{Disc}(f) = \det(Q(\alpha_1, \ldots, \alpha_n))^2 = \mathrm{Disc}_{\{1,\alpha,\ldots,\alpha^{n-1}\}}(\mathrm{tr})$$

Proposition 1.4. *Let $F(\alpha)/F$ be separable of degree n, and let f be the minimum polynomial of α. Then*

$$\mathrm{Disc}(f) = (-1)^{n(n-1)/2} N_{E/F}(f'(\alpha))/$$

Proof. Let $f(r) = \prod_{i=1}^n (t-\alpha_i)$. Then $f'(t) - \sum_{i=1}^n \prod_{j\neq i}(t-\alpha_j)$, and $f'(\alpha_i) = \prod_{j\neq i}(\alpha_i - \alpha_j)$. Then

$$\begin{aligned} N_{E/F}(f'(\alpha_i)) &= \prod_{j=1}^n \sigma_j(\prod_{j\neq i}(\alpha_i - \alpha_j)) \\ &= \prod_{(i,j), i\neq j} (\alpha_i - \alpha_j) \\ &= (-1)^{n(n-1)/2} \prod_{1\leq i<j\leq n} (\alpha_j - \alpha_i) \\ &= (-1)^{n(n-1)/2}\, \mathrm{Disc}(f). \end{aligned}$$ □

Corollary 1.1. *Let E/F be separable. The discriminant of the trace form is nonzero.*

Proof. Write $E = F(\alpha)$. Write $\beta = \{1, \alpha, \alpha^n\}$. Let f be the minimum polynomial of α. Then

$$\mathrm{Disc}_\beta(\mathrm{tr}) = \mathrm{Disc}(f) = \pm N_{E/F}(f'(\alpha)) \neq 0.$$ □